Quantum Correlations in Multipartite States. Study Based on the Wootters-Mermin Theorem

F. Herbut

Abstract
Decomposition of any N-partite state (density operator) into clusters (that do not overlap) is studied in detail with a view to learn as much as possible about the correlations implied by the state. The Wootters-Mermin theorem, stating that the totality of all strings of cluster events (projectors) determines the state in any finite- or infinite-dimensional state space, is a slightly sharpened and generalized form of the original results of Wootters and Mermin. The theorem is applied to tensor factorization of the state into states of clusters (uncorrelated decomposition) and it is shown that a finest uncorrelated decomposition always exists, and that its coarsenings and only they are other possible uncorrelated cluster decompositions. Distant effects within homogeneous cluster states, which are, by definition, the tensor factors in the finest uncorrelated decomposition, are discussed. The entire study is viewed by the author as a possible further elaboration of Mermin's Ithaca program.

Keywords Correlations. Multipartite states. Coincidence of subsystem events. Tensor factorization of state.

1. Introduction

Mermin's "Ithaca interpretation" (I'd rather call it the "Ithaca program") [1] was written in a very inspiring way. It postulated that *probability* must have a (yet unknown) meaning for the individual quantum system, and that it must be the basic notion of quantum mechanics.

In the standard formalism of quantum mechanics the basic concept is the state (a density operator) of the system. We know from Gleason's celebrated

F. Herbut (mail)
Serbian Academy of Sciences and Arts, Knez Mihajlova 35, 11000 Belgrade, Serbia
e-mail: fedorh@infosky.net and from USA etc. fedorh@mi.sanu.ac.rs

theorem [2] that 'state' and the entirety of probabilities are equivalent. (See perhaps the author's discussion of the assumptions of Gleason's theorem in [3], subsections 4.3 and 4.5.) Thus, the probabilities are the *observable* aspect of the state, and the latter is, as Mermin puts it, an 'encapsulation' of the entirety of the former.

In the case of *composite systems*, e. g., two particles far apart, only subsystem (single-particle) observables can be measured at best in coincidence. First Wootters [4], then Mermin [1] have called attention to the important fact that any composite-system state is determined by averages of strings of subsystem observables. Both authors gave nice proofs of their claims. Wootters' theory is critically reproduced in Section 2. Mermin's theorem is reproduced in Section 3 with two slight elaborations: It is shown that Mermin's subsystem observables can be restricted to subsystem elementary events (ray projectors), and that the theorem is valid also in infinite-dimensional (separable) Hilbert spaces.

Naturally, one wants to know what kind of states of composite systems one can encounter and what the correlations can do in them. Several researchers [5], [6], [7], [8] approached this problem in an inductive way: by constructing concrete examples, and then drawing conclusions from them. On account of the results of Wootters, of Mermin, and of these concrete examples a correlations structure has begun to show through a mist of insufficient comprehension.

In this article the approach is reversed: We derive a general theory of composite-system states in a deductive way leaning heavily on the Wootters-Mermin theorem. Then we discuss the mentioned examples of composite-system states.

Seevinck [8] seems to have been carried away by his result. He claims to have shown that Mermin's Ithaca mantra "the correlations, not the correlata" is disproved because, as he maintains to have shown, the correlations lack local reality.

'Reality' is a very serious question. Some of us feel a kind of piety for 'reality'. I cannot put this better than John S. Bell [9], though his remark was aimed against the instrumentalist approach, which shrinks reality to correlating successive instrumental readings:

> "... experiment is a tool. The aim remains: to understand the world. To restrict quantum mechanics to be exclusively about

piddling laboratory operations is to betray the great enterprise. A serious formulation will not exclude the big world outside the laboratory."

Bell, a theorist, did not even mention theoretical research efforts; his priority was the experiment. But along with his "piddling laboratory operations" one can put 'piddling probability calculations'.

I cannot imagine how can anybody doubt the *reality* of *observable* elements of nature. I would add two requirements to the "Ithaca program" [1] to give 'reality' the position that it deserves (though Mermin might, perhaps, not agree):

a) If a quantum-mechanical entity (simple or complex) is observable in the laboratory, then it is real.

b) If the quantum-mechanical formalism establishes a natural equivalence of two quantum-mechanical entities (simple or complex) without any arbitrariness, and if one of them is real, then so is the other.

Requirement (b) is modeled on the example of Gleason's theorem [2], and reality of the quantum state (density operator) is the first and most fundamental application of this requirement.

2. The Theory of Wootters

A Hermitian $M \times M$ matrix can be specified by giving the M real diagonal elements and the $M(M-1)/2$ complex elements above the diagonal. If the matrix has trace 1, then it contains $(M-1)+M(M-1)=M^2-1$ independent real numbers.

If one performs M^2-1 non-trivial yes-no measurements, then the state of the system is determined. We elaborate this claim of Wootters [4] in the Appendix.

Wootters came up with the fascinating observation [4] that the expounded (M^2-1)-arithmetic result *implies* that an n-partite composite system can be determined by yes-no subsystem coincidence measurements, in particular, by those that determine the subsystem states. Let us see this in more detail.

Let M_k be the dimension of the state space of the k-th subsystem (Wootters' theory was confined to finite-dimensional state spaces). Then, according to the above arithmetical claim, one needs $\left(\prod_{k=1}^n M_k\right)^2 - 1$ yes-no measurements, i. e., measurements of non-trivial events (projectors) to

specify the composite-system state. To evaluate the state of the k-th subsystem, one needs $M_k^2 - 1$ corresponding non-trivial subsystem measurements. Consider now strings of n of these same subsystem measurements including the certain events (the identity operators) any number of times. The number of these strings can be obtained by joining the certain event to each of the $(M_k^2 - 1)$ yes-no measurements, making a set of M_k^2 events, and by multiplying these numbers: $\prod_{k=1}^n M_k^2$. If we now subtract the coincidence of certain events in all subsystems, i. e., the certain event for the composite system (it gives no information), we obtain $\left(\prod_{k=1}^n M_k\right)^2 - 1$, the number of non-trivial yes-no measurements needed to determine the composite-system state (density matrix). (Note that the composite-system event is non-trivial if at least one of the subsystem events in the string is non-trivial.)

Wootters himself comments on this amazing result saying: "In this sense quantum mechanics uses its information economically."

3. The Wootters-Mermin Theorem

As it was mentioned in the Introduction, the fundamental connection between probabilities of *all* events (projectors) on the one hand and states of quantum systems (density operators) on the other is established by the well-known theorem of Gleason [2] (see also [3], subsections 4.3 and 4.5). In case of composite systems subsystems may be spatially apart from each other, and two-subsystem or more-subsystem events cannot be realized in measurement. Hence, if the requirement of 'all events' in Gleason's theorem could not be confined to 'coincidences of subsystem events' for composite systems, the state concept, and all first-quantization quantum mechanics would break down (into useless fiction).

This is where the Wootters-Mermin theorem (to be stated and proved) saves quantum mechanics supplementing Gleason's theorem in a satisfactory way.

The basic ingredients of the theorem are the mentioned coincidences of subsystem events apparently first discovered by Wootters [4], and aftewards independently discovered by Mermin [1]. But the approach taken below in the formulation of the theorem follows Mermin's ideas, not those of Wootters, because the latter's arithmetic does not allow extension to infinity.

We assume that the state space is finite or infinite dimensional. More precisely, we have a complex, separable Hilbert space. Further, we have in mind an N-partite, e. g., an N-particle quantum system. For simplicity, we'll use the language of particles.

Definition 1. We envisage the set $\{1, 2, \ldots, N\}$ of all particles in the system decomposed into n, $1 \leq n \leq N$, non-overlapping classes the union of which gives the entire set. We call the classes *clusters*, and the decomposition a *cluster decomposition* (CD). If $n = 1$, then the decomposition is said to be trivial. If $n = N$, then the CD is maximal.

Now a slightly sharpened and generalized form of the results of Wootters and Mermin, is presented as a theorem. It is called the Wootters-Mermin theorem, but also the *CD theorem.*

Theorem 1. Let us consider a *composite system* in an arbitrary given state ρ and an arbitrary given cluster decomposition (cf Definition 1). The state is *uniquely determined* by the probabilities of *all* subsystem coincidences $P_1 \otimes P_2 \otimes \ldots \otimes P_n$. Here each event P_k, $k = 1, 2, \ldots, n$ (projector in $\mathcal{H}_k$, the state space of the k-th cluster) is an elementary event (a ray projector) or the certain event (the identity operator).

We will often write $P_1 P_2 \ldots P_n$ instead of $P_1 \otimes P_2 \otimes \ldots \otimes P_n$ meaning by P_k actually $I_1 \otimes \ldots \otimes P_k \otimes \ldots \otimes I_n$ for $1 < k < n$ in this context, where I_k is the identity operator in $\mathcal{H}_k$. (For $k = 1$ and $k = n$ the required modification is obvious.)

Proof. Let Q be a projector (event) in the state space of the given composite system. Then, according to the well-known theorem of Gleason [2], the totality of all events Q determines ρ uniquely via the quantum-mechanical probability formula

$$\mathrm{tr}(Q\rho). \tag{1}$$

We make restriction to finite-trace projectors Q because they, in contrast to infinite-trace ones, do belong to the Hilbert space of Hilbert-Schmidt (HS) operators. This makes the proof rigorous. Finite-trace projectors are sufficient for the proof because infinite-trace projectors are expressible as

limeses of finite-trace ones, and probability is continuous. Thus, the infinite-trace-projector probabilities are implied by the finite-trace-projector ones.

Let $\{|\, m_k\rangle_k : m_k = 1, 2, \ldots\}$ be arbitrary complete orthonormal bases in the state spaces $\mathcal{H}_k$, $k = 1, 2, \ldots, n$ such that the quantum numbers are ordered (like the natural numbers). Every mentioned projector Q can be represented in this basis because $\prod_{i=1}^{\otimes N} \mathcal{H}_i = \prod_{k=1}^{\otimes n} \mathcal{H}_k$ (the equality actually means isomorphism). To write the representation in operator form, one introduces the corresponding dyads. Then the expansion reads

$$Q = \sum_{m_1} \sum_{m'_1} \sum_{m_2} \sum_{m'_2} \ldots \sum_{m_n} \sum_{m'_n} \langle m_1 |_1 \, \langle m_2 |_2 \ldots \langle m_n |_n \, Q \, | m'_1 \rangle_1 \, | m'_2 \rangle_2 \ldots | m'_n \rangle_n \times$$

$$| m_1 \rangle_1 \langle m'_1 |_1 | m_2 \rangle_2 \langle m'_2 |_2 \ldots | m_n \rangle_n \langle m'_n |_n \, . \tag{2}$$

To rid ourselves of the off-diagonal dyads $|\, m_k\rangle\langle m'_k\,|$, $m_k < m'_k$, $k = 1, 2, \ldots, n$, and replace them by ray projectors, we define

$$\forall (m_k < m'_k): \quad | m_k, m'_k, 1\rangle \equiv (1/2)^{1/2} (| m_k\rangle + | m'_k\rangle),$$

$$| m_k, m'_k, 2\rangle \equiv (1/2)^{1/2} (| m_k\rangle - i \, | m'_k\rangle).$$

Inverting these definitions, one obtains

$$\forall (m_k < m'_k): \quad | m_k\rangle\langle m'_k | = | m_k, m'_k, 1\rangle\langle m_k, m'_k, 1| - i \, | m_k, m'_k, 2\rangle\langle m_k, m'_k, 2| +$$

$$[(i-1)/2] \, | m_k\rangle\langle m_k | + [(i-1)/2] \, | m'_k\rangle\langle m'_k | \, . \tag{3}$$

Naturally, $| m'_k\rangle\langle m_k | = (| m_k\rangle\langle m'_k |)^\dagger$.

Next, we replace each off-diagonal dyad by the linear combination of 4 ray projectors according to (3) or its adjoint. One should note that, while (2), if a series, is absolutely convergent, hence the order can be changed by an arbitrary (infinite) permutation. This is no longer true when the expansion is exclusively in diagonal dyads, which are ray projectors. (The order still may be permuted if the 4 dyads introduced by (3) or its adjoint are kept together for all $k \neq k'$.) The above 'diagonalization' procedure leading to (3) is not unique.

All that remains to be done is to substitute Q in (1) by its expansion (2) in which the off-diagonal dyads have been replaced by ray projectors according to (3) or its adjoint. The sums (series), along with the numbers can be taken outside the trace due to the linearity (and continuity) of the trace. Then we have linear combinations (possibly infinite ones) of probabilities of

coincidences of elementary cluster-events (ray projectors). $\square$

One should have in mind that the scalar product in the Hilbert space of two HS operators A and B is $(A,B) \equiv \mathrm{tr}\Big(A^\dagger B\Big)$.

The proof of Mermin's theorem presented actually confines the strings of subsystem ray projectors far more than stated in the theorem. (They are all generated from one fixed basis.) But we will not utilize this stronger form. Actually, we'll often make use of a form that is even weaker than the formulation of the theorem: we'll use any subsystem events (not just elementary ones).

4. Immediate Consequences

One may object that the Wootters-Mermin theorem does not give a practical way how to evaluate ρ from the coincidence probabilities. Neither does Gleason's theorem. I see the former as an important elaboration of Gleason's theorem in the case of composite systems. The fundamental significance of both lies in their generality.

Next, one wonders what correlations are. This is a very elusive concept. The only case that I know when one can put one's finger on an entity expressing the correlations is the case of bipartite state vectors, where the (antiunitary) correlation operator 'carries' all correlations. (More about this below, in Lemma 1.)

As it is in Gleason's case, where each positive-probability event 'probes' the state, in composite systems the strings of subsystem events 'probe' the state, and *ipso facto* they 'probe' the correlations implied by the state. In lack of a general definition of correlations, there is a natural way how to define sort of 'part' of the correlations that a given string of subsystem events 'sees' probing the state.

In classical probability theory it is well understood that events are uncorrelated if the coincidence probability equals the product of the separate probabilities of the events. Classical probability theory is relevant because the projectors in a string of subsystem events always commute with each other. Hence, it seems natural to make the following definition *of the correlations that a given string of subsystem events 'sees' probing the state.*

Definition 2. Let ρ be a state of a composite system of N particles, and let a CD be given that breaks up the set of all particles into n clusters (cf Definition 1). Let, further, $P_1P_2\dots P_n$ be a given string of subsystem events - each factor being a subsystem projector possibly the identity operator. Then the correlations that the string 'sees' in ρ is *the absolute value of the difference between the coincidence probability and the product of subsystem-event probabilities*:

$$\Big|\mathrm{tr}\Big(\rho\prod_{k=1}^{\otimes n}P_k\Big)-\prod_{k=1}^{n}\mathrm{tr}(\rho P_k)\Big|. \tag{4}$$

The subsystem events are *uncorrelated* if the string 'sees' correlations that are zero; otherwise they are *correlated*.

Corollary 1. If the string in Definition 2 contains a zero-probability subsystem event, then the string 'sees' no correlations.

Proof. Let, e. g., $\mathrm{tr}(\rho P_1)=0$. This implies $P_1\rho P_1=0$ (because $\mathrm{tr}(P_1\rho P_1)=0$, and a positive operator can have zero trace only if it is zero itself). Then $\mathrm{tr}\Big(\rho\prod_{k=1}^{\otimes n}P_k\Big)=\mathrm{tr}\Big((P_1\rho P_1)\prod_{k=1}^{n}P_k\Big)=0$, and analogously $\prod_{k=1}^{n}\Big(\mathrm{tr}(\rho P_k)\Big)=0$ (cf Definition 2). $\Box$

Since zero-probability subsystem events disable any string in which they appear to 'see' correlations, it is best to avoid them.

Positive-probability events need not coincide, i.e., one can have, e. g., $\mathrm{tr}\Big(\rho_1P_1\Big)>0<\mathrm{tr}\Big(\rho_2P_2\Big)$, and $\mathrm{tr}\Big(\rho_{12}(P_2\otimes P_2)\Big)=0$. Example: $P_1\equiv|+\rangle_1\langle+|_1$, and $P_2\equiv|+\rangle_2\;|+\rangle_2$ in the singlet state.

If one does not take the absolute value in (4), then a string of subsystem events can 'see' *increase* or *decrease* of coincidence probability with respect to the product of subsystem probabilities. The example last mentioned involves decrease. The example $P_1\equiv|+\rangle_1\langle+|_1$, and $P_2\equiv|-\rangle_2\langle-|_2$ in the singlet state involves increase.

In the special case of a *bipartite state vector* $\mid\Psi\rangle_{12}$, the (antiunitary) *correlation operator* U_a , implied by the state vector, is the carrier of the entire correlations [10], [3]. Hence, one can derive the correlations (4) 'seen' by a given string of subsystem events P_1,P_2 . This is done in the next

lemma.

Lemma 1. Let $\mid \Psi\rangle_{12}$ be an arbitrary bipartite state vector, and let P_1, P_2 be arbitrary subsystem events (projectors). Then the correlations 'seen' by these events are

$$\Big|\Big(\sum_q{}^{'} \langle q\mid_1 \rho_1^{1/2} U_a^\dagger P_2 U_a \rho_1^{1/2} \mid q\rangle_1\Big) \quad - \quad \Big(\mathrm{tr}(\rho_1 P_1)\Big)\mathrm{tr}(\rho_2 P_2)\Big|,$$

where $\{\mid q\rangle_1 : \forall q\}$ is a complete orthonormal basis in $\mathcal{H}_1$ such that a subset of it spans the range of P_1 : $P_1 = \sum_q' \mid q\rangle_1\langle q\mid_1$, the prim on the sum denoting that one sums only over the mentioned subset. Naturally, $\rho_i \equiv \mathrm{tr}_j\Big(\mid \Psi\rangle_{12}\langle\Psi\mid_{12}\Big),\ i,j = 1,2,\ i \neq j$.

One should note that while U_a maps the range of ρ_1 onto that of ρ_2 , its adjoint $U_a^\dagger$, equalling its inverse U_a^{-1} , maps the latter range onto the former. Mathematicians would write, e. g., $U_a \circ \rho_1^{1/2}$ etc. $'\circ'$ meaning "after" because the operators do not act in one and the same space.

Proof. Any bipartite state vector can be expanded in any complete orthonormal first-subsystem basis, and the (generalized) expansion coefficients are the images of the corresponding basis vectors by the (antilinear) operator $U_a\rho_1^{1/2}$, which are vectors in $\mathcal{H}_2$:

$$\mid\Psi\rangle_{12} = \sum_q \Big[\mid q\rangle_1 \Big(U_a\rho_1^{1/2} \mid q\rangle_1\Big)_2\Big]$$

[10], [3] (section 2). The rest is standard evaluation:

$$\mathrm{tr}\Big(\mid\Psi\rangle_{12}\langle\Psi\mid_{12} P_1 P_2\Big) = \langle\Psi\mid_{12} \Big(\sum_q{}^{'} \mid q\rangle_1\langle q\mid_1 P_2\Big) \mid\Psi\rangle_{12}.$$

Substitution of the expanded form of the state vector in conjunction with (4) leads to the claimed result. $\square$

Though the correlations in an N-partite state ρ are, in general, an evasive entity, their measure, showing how much of them there is, is given by many authors in various forms. The present author is partial to the von Neumann entropy, and the *correlation information* $I_{12\ldots N}$ as a measure of correlations that follows from it.

Let a CD be given (cf Definition 1). On account of the universal law of subadditivity of entropy, one has the definitions:

$$I_{12\ldots N} \equiv \Big(\sum_{i=1}^{N} S_i\Big) - S_{12\ldots N}, \tag{5a}$$

where $I_{12\ldots N}$ is the correlation information in ρ , S_i is the von Neumann entropy of the i-th particle and $S_{12\ldots N}$ is the von Neumann entropy of the entire system. Further,

$$I_{C_k} \equiv \sum_{i\in C_k} S_i - S_{C_k} \quad k = 1, 2, \ldots, n, \tag{5b}$$

where "C_k" denotes the k-th cluster, I_{C_k} denotes its correlation information, and S_{C_k} is its entropy; finally,

$$I_{\sqcap} \equiv \sum_{k=1}^{n} S_{C_k} - S_{12\ldots N} \tag{5c}$$

is the among-the-clusters correlation information.

The following theorem follows easily (cf Theorem 1 in [11]):

$$I_{12\ldots N} = \sum_{k=1}^{n} I_{C_k} \ + \ I_{\sqcap}. \tag{5d}$$

In words: The total correlation information is the sum of the within-the-cluster ones, summed over all clusters, and the among-the-cluster one.

5 . The uncorrelated cluster decompositions

Definition 3. We call any cluster decomposition (cf Definition 1) that implies tensor factorization of the state of the composite system into the states (reduced density operators) of the subsystems

$$\rho = \prod_{k=1}^{\otimes n} \rho_k \tag{6}$$

an *uncorrelated cluster decomposition* (UCD).

Now we formulate and prove a basic consequence of the Wootters-Mermin theorem, the *UCD theorem*.

Theorem 2. One is dealing with an uncorrelated cluster decomposition *if and only if every string* of subsystem events $P_1 P_2 \dots P_n$ is *uncorrelated*:

$$\mathrm{tr}(\rho P_1 \dots P_n) = \prod_{k=1}^{n} \mathrm{tr}(\rho_k P_k) \tag{7}$$

(cf (4)). Equivalently, if there is a string of subsystem events that 'sees' correlations, then and only then one is not dealing with a UCD.

Proof. *Necessity.* Assuming the tensor factorization (6), one has

$$\mathrm{tr}\Big((\prod_{k=1}^{\otimes n} \rho_k)(\prod_{k=1}^{\otimes n} P_k) \Big) = \prod_{k=1}^{n} \mathrm{tr}(\rho_k P_k). \tag{8}$$

Sufficiency. Let (8) hold true for all corresponding strings. Since, according to the Wootters-Mermin theorem, they determine a unique composite-system state, it is obviously $\prod_{k=1}^{\otimes n} \rho_k$. $\Box$

Definition 4. If one has two cluster decompositions of the system considered, one says that the latter decomposition is a *coarsening* of the former if each class in the latter consists of one or several classes of the former. One also says that the former decomposition is a refinement of the latter. The corresponding adjectives are: 'coarser' and 'finer'.

If one cluster decomposition is a coarsening of another, then the former CD is, obviously, also a decomposition into classes of the set of clusters of the latter, finer cluster decomposition.

Now we formulate the *FUCD theorem*, the basic result of this investigation.

Theorem 3. A) For every composite-system state ρ , there exists, in a *unique way, a finest uncorrelated cluster decomposition*, i. e., one that implies (6) and that is such that every other UCD is its coarsening.

B) One has an uncorrelated cluster decomposition *if and only if* it is a *coarsening* of the finest cluster decomposition.

Before we prove the theorem, we need some auxiliary insight.

Lemma 2. Every cluster decomposition of $\{1,2,\dots,N\}$ *induces* a decomposition into subclusters *in every subset* of $\{1,2,\dots,N\}$ by requiring that two particles within the subset belong to the same subcluster if they belong to the same cluster in $\{1,2,\dots,N\}$.

Proof is obvious. $\Box$

Lemma 3. Every uncorrelated cluster decomposition induces (cf Lemma 2) in every subset of $\{1,2,\dots,N\}$ an uncorrelated subcluster decomposition.

Proof. Let an uncorrelated CD (cf Definition 3) and a subset of $\{1,2,\dots,N\}$ be given. Let, further, $P_1P_2\dots P_n$ be the product of arbitrary projectors on the induced subclusters in the subset. (If the subcluster corresponding to an index $1 \le k \le n$ is an empty set, then we put $P_k \equiv I$, the identity operator in $\mathcal{H}_{12\dots N}$.) The string of events 'sees' the following correlations (cf (4))

$$\Big|\mathrm{tr}\Big(\rho(\prod_{k=1}^{n} P_k)\Big) - \prod_{k=1}^{n}\mathrm{tr}\Big(\rho_k P_k\Big)\Big|,$$

where ρ_k is the state (reduced density operator) of the k-th subcluster.

One has

$$\mathrm{tr}\Big(\rho_k P_k\Big) = \mathrm{tr}\Big(\rho'_k P_k\Big), \quad k = 1,2,\dots,n, \tag{9}$$

where ρ'_k is the state of the cluster whose intersection with the given subset the subcluster is, because one obtains the lhs from the rhs by partial tracing. (Note that $P_k,\ k = 1,2,\dots,n$, are subcluster events). Hence, the string 'sees' the same correlations in the subclusters as in the clusters, where, according to the UCD theorem (Theorem 2), no correlations are 'seen'. Thus, according to the same theorem, the subclusters are in uncorrelated states. $\Box$

Next, we realize that *continuation* of a UCD is a UCD. More precisely, one has the following claim.

Lemma 4. If each cluster in a given uncorrelated cluster decomposition is further decomposed in an uncorrelated way into subclusters, then the entire decomposition of $\{1,2,\dots,N\}$ into subclusters is an uncorrelated

cluster decomposition.

Proof becomes evident when one substitutes each ρ_k in $\rho = \prod_{k=1}^{\otimes n} \rho_k$ (cf (6)) by its corresponding tensor factorized form. $\Box$

Lemma 5. The intersection of two uncorrelated cluster decompositions is a UCD.

Proof. If a UCD is intersected with another UCD, then, according to Lemma 2, each cluster of the former is decomposed into subclusters. Further, according to Lemma 3, the decompositions are uncorrelated. Finally, according to Lemma 4, the decomposition of $\{1, 2, \ldots, N\}$ into the subclusters (intersections of clusters) is a UCD. $\Box$

Lemma 6. The intersection of any number of uncorrelated cluster decompositions is a UCD.

Proof. Let us have L UCD's, $2 < L < \infty$. We order them, in an arbitrary but fixed way, into a sequence. The intersection can be obtained in a stepwise way by intersecting the first UCD with the second, the result of this with the third etc. We apply Lemma 5 at each step. $\Box$

We need just one more auxiliary result.

Lemma 7. Every coarsening (cf Definition 4) of a UCD is a UCD.

Proof becomes evident if one makes an isomorphic transition from $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_N$ to an N-particle Hilbert space ordered according to the two given CD's: Consecutive groups of particles correspond (in an arbitrary way) to clusters of the first CD, and groups of clusters according to the coarsening are also consecutive. Then it is seen that substitution of the groups of states in (6), rewritten according to the new ordering, by the states corresponding to the coarsening does not change the tensor-product structure of the relation. $\Box$

Proof of the FUCD theorem.

A) According to Lemma 6, the intersection of all UCD's is a UCD. It is obvious that it is the finest of all of them.

B) It is obvious from claim A) that every UCD must be a coarsening of

the finest one. Conversely, that every coarsening of the finest UCD is a UCD is a consequence of Lemma 7. □

Definition 5. We say that the state of a subsystem is *homogeneous* if it appears among the tensor factors in the finest uncorrelated decomposition (cf (6)). Otherwise it is said to be *heterogeneous*.

Next, we study states of homogeneous clusters.

6. Dynamical study.

Influence of subsystem measurement and unitary evolution on the opposite subsystem in a bipartite system is going to be investigated.

6.A Ideal subsystem measurement.

We begin with the simple, textbook case of measurement. We have a discrete decomposition of the identity $I = \sum_{q=1}^{Q} P_1^q$, Q finite or infinite ($\forall q, q' :\ P_1^q P_1^{q'} = \delta_{q,q'} P_1^q = \delta_{q,q'} (P_1^q)^\dagger$).

Let us consider a bipartite system in a state ρ_{12} . Let us, further, consider an *ideal first-subsystem measurement* in the non-selective version (when one deals with the entire ensemble) ascertaining which of the events P_1^q will occur on each element of the ensemble. When this measurement is performed, then the state (reduced density operator) of subsystem 2 does not change, but becomes decomposed (distant state decomposition)

$$\rho_2 = \sum_{q=1}^{Q} \Big(\mathrm{tr}(\rho_{12} P_1^q)\Big) \rho_2 \Big\{\rho_{12}, P_1^q\Big\}, \qquad (10a),$$

where the probabilities $\mathrm{tr}(\rho_{12} P_1^q)$ are the weights, and $\rho_2 \Big\{\rho_{12}, P_1^q\Big\}$ are the admixed states defined as follows

$$\forall q,\ \mathrm{tr}(\rho_{12} P_1^q) > 0 : \quad \rho_2 \Big\{\rho_{12}, P_1^q\Big\} \equiv \mathrm{tr}_1(\rho_{12} P_1^q) \Big/ \mathrm{tr}(\rho_{12} P_1^q), \qquad (10b)$$

where "tr_1" denotes the partial trace over $\mathcal{H}_1$.

To prove claims (10a,b), remember that ideal measurement in its non-selective version, by definition, gives a change of state according to the Lüders formula [12], [13], [14]:

$$\rho_{12} \quad \rightarrow \quad \sum_{q=1}^{Q} P_1^q \rho_{12} P_1^q. \tag{11}$$

The second-subsystem state is obtained by tracing out subsystem 1 (and taking into account that P_1^q commutes under the partial trace tr_1 with the other factor $(\rho_{12}P_1^q)$ and that P_1^q is idempotent). Thus one obtains (10a) with (10b).

We call decomposition (10a) of the state of subsystem 2 'distant' because the measuring apparatus is assumed to interact only with subsystem 1 , and not at all with subsystem 2 . Therefore, any influence of this measurement on subsystem 2 is due exclusively to the correlations between subsystems 1 and 2 . The term 'distant' should capture this circumstance (in analogy with the case when there is large spatial distance between the subsystems).

Distant state decomposition (10a) is just one of the countless mathematically possible decompositions of ρ_2 . Its *physical meaning* comes to the fore in the selective aspect of the same measurement.

When we consider the same measurement in the selective version, i. e., when the state of individual systems, elements in the ensemble, are the object of description, then the composite system undergoes the change

$$\rho_{12} \quad \rightarrow \quad P_1^q \rho_{12} P_1^q \Big/ \mathrm{tr}(\rho_{12}P_1^q) \tag{12a}$$

(unless P_1^q is a probability-zero event). As a consequence of this ideal occurrence of P_1^q , the state of the second subsystem is subject to the (violent) change

$$\rho_2 \quad \rightarrow \quad \rho_2\Big\{\rho_{12}, P_1^q\Big\} \tag{12b}$$

(cf(10b)).

If the bipartite state is uncorrelated, i. e., if $\rho_{12} = \rho_1 \otimes \rho_2$, then

$$\forall q, \;\; \mathrm{tr}(\rho_{12}P_1^q) > 0 : \quad \rho_2\{\rho_{12}, P_1^q\} = \rho_2, \tag{13}$$

(as it is obvious from (10b)), and the distant state decomposition (10a) is trivial. In other words, if two subsystems are uncorrelated, ideal measurement in one of them cannot influence the other.

Thus, *the finest uncorrelated decomposition discloses the boundaries where the distant influence of subsystem measurement stops.*

There is a serious objection to the importance of the conclusion just reached. Namely, ideal measurement is overidealized; it is hard to make use of it in the laboratory. More general measurements are the realistic ones [15]. One wonders if the boundaries of uncorrelatedness stop also the influence of more realistic measurements.

6.B Realistic measurements.

In realistic measurements the change of state is far more complicated than in the ideal case [15]. It is therefore desirable to avoid it. We make a plausible assumption, and then resort to classical probability theory.

Assumption. If we have a bipartite state ρ_{12} and a coincidence of subsystem events $P_1^qP_2$ (cf Subsection 6.A for the notation) in an arbitrary measurement, then the coincidence probability *equals* the product of the probability of the event P_1^q and the probability of P_2 in the state of subsystem 2 that comes about as a result of the ideal occurrence of P_1^q . Naturally, we assume that P_2 occurs immediately after P_1^q . More precisely, we take the limes when the time interval between these two occurrences goes to zero.

The Assumption amounts to equating coincidence with the conditional probability formula in classical probability theory. One must keep in mind Mermin's warning [1] that one must be very cautious when interpreting conditional probability in quantum mechanics, because the condition, a correlatum in Mermin's terms, does not exist (except in the trivial case when the conditional probability equals the original one).

The classical formula is easily understood in terms of relative frequencies in the measurement. Let M_{12} be the frequency of the coincidences $P_1^qP_2$, M the total number of occurrences in the measurement, M_1 the frequency of P_1^q , and, finally, M_2 that of P_2 . Then, as well known, the coincidence probability is the limes of $M_{12}\Big/M$ when M goes to infinity.

One can write $M_{12}\Big/M=(M_1\Big/M)(M_{12}\Big/M_1)$ (unless $M_1=0$). Taking the limes separately of each factor on the rhs, one comes to the mentioned classical conditional probability formula. Thus, the 'condition' is well defined: we have in mind the cases when P_1^q occurs, then it is a given correlatum, and it can serve the purpose of a condition.

In the quantum-mechanical formalism the conditional probability argument goes as follows.

$$\mathrm{tr}(\rho_{12}P_1^qP_2)=\Big(\mathrm{tr}(\rho_{12}P_1^q)\Big)\Big[\mathrm{tr}\Big(\rho_2\{\rho_{12}P_1^q\}P_2\Big)\Big] \tag{14}$$

(cf (10b)).

The argument presented establishes the fact that the distantly prepared state of subsystem 2 , when P_1^q occurs in realistic measurement, is the same as in ideal measurement. Hence also the important conclusion that *boundaries of uncorrelatedness stop measurement influence* is valid in the general case.

6.C Subsystem interaction has no distant influence

The claim in the title of the subsection is proved by the following elementary argument.

Let ρ_{012} be a tripartite system such that $\rho_{12}\equiv\mathrm{tr}_0\rho_{012}$ is a bipartite system under investigation. These entities apply to an initial moment: $\rho_{012}\equiv\rho_{012}(t=0)$.

We assume that subsystems 0 and 1 interact in an arbitrary way, but that subsystem (0+1) does not interact with the (hence distant) subsystem 2 . We express this by the tensor product of unitary (dynamical evolution) operators

$$\rho_{012}\quad\rightarrow\quad\rho_{012}(t)\equiv\Big(U_{01}(t)\otimes U_2(t)\Big)\rho_{012}\Big(U_{01}(t)^\dagger\otimes U_2(t)^\dagger\Big).$$

Then we have

$$\rho_2(t)\equiv\mathrm{tr}_{01}\Big(\rho_{012}(t)\Big)=\mathrm{tr}_{01}\Big[\Big(U_{01}(t)\otimes U_2(t)\Big)\rho_{012}\Big(U_{01}(t)^\dagger\otimes U_2(t)^\dagger\Big)\Big]=$$

$$U_2(t)\Big[\mathrm{tr}_{01}\Big(U_{01}(t)\rho_{012}U_{01}(t)^\dagger\Big)\Big]U_2(t)^\dagger=U_2(t)\Big[\mathrm{tr}_{01}\Big(\rho_{012}U_{01}(t)^\dagger U_{01}(t)\Big)\Big]U_2(t)^\dagger=$$

$$U_2(t)\Big(\mathrm{tr}_{01}(\rho_{012})\Big)U_2(t)^\dagger=U_2(t)\rho_2U_2(t)^\dagger.$$

This known fact can be put so that whatever goes on dynamically with the state of subsystem 1 when no dynamical influence is exerted on subsystem 2, it *does not influence the latter via correlations either.* This general claim is derived after the measurement influences of the preceding two subsections because of the striking *contradiction.* Namely, one expects measurement to be a special case of dynamical evolution. (This is the so-called paradox of the quantum theory of measurement.)

The way I see it, there are two basic schools of thought in foundational quantum mechanics. The first stipulates that unitary dynamical evolution is not universal; altered dynamics applies to measurement [16].

The second school of thought sticks to exclusively unitary dynamics, but it abandons the 'prejudice' of absolute properties to which we are used from classical physics, particularly from special relativity theory. Jordan [5], in a skilful variation of Hardy's approach to Bell's theorem [17], proves that the assumption of local and real properties - which is the same as absolute properties - of the famous EPR argument [18] is in contradiction with quantum mechanics.

To avoid the paradox, it takes some kind of relative-state view in the spirit of Everett [19], like, e. g., the relative-reality-of-unitarily-evolving-state (RRUES) view as it was expounded in recent quantum-mechanical discussions of the delayed-choice erasure experiments of Scully et al. [20], [21]. We will resume this point of view in the next section after we present a beautiful EPR-type entanglement case, introduced in the after-Mermin investigations by Cabello.

7. EPR-type entanglement

Cabello [6] suggested to consider a quadri-partite purely-spin state vector that is the tensor product of two singlet states:

$$|\Psi\rangle_{1234} \equiv \Big((1/2)^{1/2}(|+\rangle_1\,|-\rangle_2 - |-\rangle_1\,|+\rangle_2)\Big) \otimes \Big((1/2)^{1/2}(|+\rangle_3\,|-\rangle_4 - |-\rangle_3\,|+\rangle_4)\Big). \tag{15}$$

In view of the FUCD theorem, it is easy to see that this state is written in its finest uncorrelated decomposition form (because the decomposition cannot be continued), and that there is no other non-trivial CD ((15) does not have a non-trivial coarsening, cf the FUCD theorem).

The subsystems $(2+3)$ and $(1+4)$, which are bipartite in their turn,

are correlated in (15). We first change the order of the tensor factors in (15) (by isomorphism) from 1234 to 2314. Then, as easily seen, (15) can be rewritten in the (isomorphic) form as a (maximally correlated) Schmidt canonical decomposition [3] (section 2).

$$|\Psi\rangle'_{2314} = (1/2)\Big[\Big(\,|-\rangle_2\,|+\rangle_3\Big)\otimes\Big(\,|+\rangle_1\,|-\rangle_4\Big) \;+\; \Big(\,|-\rangle_2\,|-\rangle_3\Big)\otimes\Big(-\,|+\rangle_1\,|+\rangle_4\Big) \;+$$

$$\Big(\,|+\rangle_2\,|+\rangle_3\Big)\otimes\Big(-\,|-\rangle_1\,|-\rangle_4\Big) \;+\; \Big(\,|+\rangle_2\,|-\rangle_3\Big)\otimes\Big(\,|-\rangle_1\,|+\rangle_4\Big)\Big]. \tag{16}$$

Since the eigenvalue $1/4$ of the reduced density operator ρ_{23} (as well as that of ρ_{14}) has fourfold degeneracy, expansion of $|\,\Psi\,\rangle'_{2314}$ in any basis in the four-dimensional space $\mathcal{H}_2\otimes\mathcal{H}_3$ gives a canonical Schmidt decomposition (cf [3], section 2). The basis in $\mathcal{H}_2\otimes\mathcal{H}_3$ in which Ψ'_{2314} is expanded in (16) is uncorrelated. Cabello rightly suggests [6] to take an opposite case, i.e., one with a basis of maximally correlated state vectors. The well-known Bell states

$$|\psi^{\pm}\rangle_{23} \equiv (1/2)^{1/2}\Big(\,|+\rangle_2\,|-\rangle_3 \;\pm\; |-\rangle_2\,|+\rangle_3\Big), \tag{17a}$$

$$|\phi^{\pm}\rangle_{23} \equiv (1/2)^{1/2}\Big(\,|+\rangle_2\,|+\rangle_3 \;\pm\; |-\rangle_2\,|-\rangle_3\Big), \tag{17b}$$

which also form an orthonormal basis in $\mathcal{H}_2\otimes\mathcal{H}_3$, are quite suitable.

The evaluation of the 'partner' in each term of a Schmidt canonical decomposition is much facilitated by the use of the (antiunitary) *correlation operator* U_a that is uniquely implied by every bipartite state vector. It is an *invariant* entity for all Schmidt canonical decompositions of a given bipartite state vector, and it maps precisely the first-tensor-factor basis vectors into the second ones, into their 'partners' in the decomposition [3] (Appendix A there).

Therefore, it is practical to start with the Schmidt canonical decomposition that is an expansion in the uncorrelated basis as (16) is, read U_a in it, and then one can immediately write down the 'partners' in any other Schmidt canonical decomposition. In this way one obtains:

$$|\Psi\rangle'_{2314} = (1/2)\Big[\,|\psi^+\rangle_{23}\otimes|\psi^+\rangle_{14} \;+\; |\psi^-\rangle_{23}\otimes\Big(-\,|\psi^-\rangle_{14}\Big) \;+$$

$$|\phi^+\rangle_{23}\otimes\Big(-\,|\phi^+\rangle_{14}\Big) \;+\; |\phi^-\rangle_{23}\otimes\Big(-\,|\phi^-\rangle_{14}\Big)\Big]. \tag{18}$$

If the subsystems are spatially sufficiently far away from each other so that one can perform subsystem measurements, which by definition must not dynamically influence the opposite (or distant) subsystem, then any Schmidt canonical decomposition has an important *physical meaning*. (In a purely spin state as (15) is, one can safely assume the feasibility of subsystem measurement because in the suppressed spatial tensor-factor part of the state the two particles can be far away from each other.)

Let us take (16). If one performs on the nearby subsystem $(2+3)$ a measurement to ascertain in which of the uncorrelated states (first tensor factors in (16)) the subsystem is, then *ipso facto*, by distant, i. e., by interaction-free measurement the distant subsystem $(1+4)$ finds itself in the uncorrelated 'partner' state. (Schrödinger [22], [23] would say that the distant subsystem is 'steered' into the 'partner' state.)

If we decide to measure on the subsystem $(2+3)$ in which of the (maximally correlated) Bell states (17a,b) it is, then, a look at (18) tells us that *ipso facto* one finds out by distant measurement in which of the same Bell states the distant subsystem $(1+4)$ is. This is EPR-type disentanglement, the heart of the famous EPR paradox [18].

Note that the two mentioned direct subsystem measurements are mutually incompatible, and usually they are considered as alternative choices. But the real random delayed-choice erasure experiment of Kim et al. [24] has shown that it is possible to perform the two mutually incompatible measurements in one and the same experiment (cf also the quantum-mechanical insight in the experiment [21]).

Also Seevinck mentions the above distant measurement of Bell states, but he views it as entanglement swapping ([8], Section 5, (i)).

EPR-type disentanglement is a striking example of what the correlations can do if a boundary appearing in a UCD does not stop it (like in the bipartite system $(1+2)+(3+4)$, cf (15), in contrast to $(2+3)+(1+4)$, cf (16)).

Returning to the second school of thought in foundational quantum mechanics, which maintains the exclusiveness of unitary evolution, mentioned in the preceding section, we can repeat the Ithaca mantra of Mermin: "The correlations, not the correlata". In the above case, both (16) and (18) simultaneously really exist in the given quadripartite state vector along with infinitely many other (also incompatible) Schmidt canonical decompositions. The 'partner' subsystem states in each term express aspects of the correlation, which is best encapsulated in the correlation operator U_a , which covers all possible Schmidt canonical decompositions. Measurements only add

new subsystems (the measuring instruments) to make a more complex multipartite state vector, but essentially nothing is changed. (See the quantum-mechanical insight in delayed-choice erasure experiments in [20] and [21].)

The "not the correlata" part of Mermin's mantra means, the way I understand it, that the tensor factors in the components of e. g. (16) or (18) or any other concrete Schmidt canonical decomposition, or rather the elementary events that they define, cannot be considered real in an absolute sense. In the standard quantum-mechanical language , they are potentialities. (This corresponds to the more usual claim that observables do not have definite values in such cases.)

In the relative-reality-of-unitarily-evolving-state (RRUES) approach, which is in the spirit of Everett [19], and which seems to be required by the exclusively unitary evolution, *the correlations*, along with the particles of which the systems are made up, appear to be *the basic building blocks of reality.*

8. Correlational isolation or being correlationally closed

Definition 6. A state $\rho_{12\dots N}$ of a system of N particles is *correlationally isolated* (from its environment) or *correlationally closed* if, whenever K particles from the environment, $1 \leq K$, are joined to the quantum-mechanical description, in the state of the enlarged system (of $N+K$ particles) the original system of N particles is uncorrelated with the K added ones:

$$\rho_{12\dots N(N+1)\dots(N+K)} = \rho_{12\dots N} \otimes \rho_{(N+1)\dots(N+K)}, \tag{19}$$

where the factors are the corresponding subsystem states (reduced density operators). Otherwise the state of the system is correlationally open or unisolated.

We state and prove now a result *on subsystem inheritance.*

Proposition. If $\rho_{12} = \rho_1 \otimes \rho_2$ is an uncorrelated state of a bipartite system, then also the state of each subsystem of subsystem 1 is uncorrelated with the state of any subsystem of subsystem 2 .

Proof follows immediately when one takes the partial traces (in the above tensor product) over the particles that do not belong to the (smaller) subsystems considered. □

Corollary 2. If the state of a system is correlationally isolated from its environment, then so is the state of its every subsystem.

Proof. The preceding proposition immediately implies Corollary 2. $\Box$

If a cluster state is homogeneous in the state ρ of the N-particle system and it is also correlationally isolated from the environment of the latter, irrespectively if so is also ρ, then we say that the state of the cluster is *absolutely homogeneous*.

It should be noted that if one considers the finest uncorrelated decomposition of a correlationally open system, each of the homogeneous subsystems can be, independently of each other, open or closed correlationally. We have seen in the preceding section what apparently devastating influence the correlations can transfer from the nearby subsystem to the distant one.

9. Comments on Seevinck's and Cabello's articles

Now we take a critical look from the point of view of the CD, UCD, and FUCD theorems of this article at some mentioned important work.

To my knowledge, Jordan [5] was the first to take a critical view of the Ithaca program [1]. It was pointed out (in the last-but-one passage of Section 6) that this author made an important contribution to understanding distant correlations. His criticism of Mermin is similar to that of Cabello, and the latter is more clearly articulated. Therefore, I wont discuss Jordan's critical attitude separately.

9.A Seevinck's article

In [8] the title reads: "The Quantum World is not Built up from Correlations". This claim should be contrasted with the FUCD theorem (Theorem 3) of this study, which says that every state of a several-particle system is the tensor product of homogeneous disjoint cluster states, i. e., of states that cannot be further tensor factorized, and that this is unique. Hence, the composite-system state is built up from homogeneous cluster states. There are no correlations between these clusters, but the cluster states contain correlations.

In the Introduction of [8] the question of anticorrelation of spin projections in the singlet state is raised and it is asked "whether or not we can think of this (anti-) correlation as a real property of the two-particle system independent of measurement". Later on it is stated that "in this letter we will demonstrate that ... no such interpretation is possible".

The singlet two-particle state, as any pure state, is a tensor factor in the state of any cluster containing the two particles (it is absolutely homogeneous in the terms of this study). There is no reason why the mentioned anticorrelation could not be viewed as a piece of reality of nature. (We return to this below.)

To my mind, the central point in Seevinck's article is his assumption of "local realism". He expresses it for a four-partite system that he views as a bipartite system the subsystems of which are, in turn, each bipartite. It reads (cf his relation (6)):

$$P_{\hat{A}\hat{B},\hat{C}\hat{D}}(ab,cd|W_0) = P^I_{\hat{A}\hat{B}}(ab|W_I)P^{II}_{\hat{C}\hat{D}}(cd|W_{II}), \tag{20}$$

where W_0 is the state (density operator) of the composite system, and $W_i,\ i = I, II$ are the states (reduced density operators) of the subsystems. Further, $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ are observables (Hermitian operators) of the four finest subsystems, and a, b, c, d are their possible eigenvalues. The lhs

$$P_{\hat{A}\hat{B},\hat{C}\hat{D}}(ab,cd|W_0) = \mathrm{tr}\Big(W_0\hat{A}\hat{B}\hat{C}\hat{D}\Big)$$

is the average value of the product of the four observables, and on the rhs we have the product of corresponding averages in the two (larger) subsystems.

Since Hermitian operators are linear combinations of their spectral projectors (if one confines oneself to observables with finite spectra), it is sufficient to restrict the four observables to projectors.

Remembering the UCD theorem (Theorem 2), one can expect (20) to be valid if the two larger subsystems are, as clusters, uncorrelated, i. e., if $W_0 = W_I \otimes W_{II}$. Otherwise, one would expect that one can choose the four projectors so that (20) is not valid.

If one finds the alleged "local realism" condition (20) not valid, this has nothing to do with "realism", only with lack of tensor factorization (lack of uncorrelatedness). It does have to do with "localness" because Seevinck uses this term as a synonym for subsystem: "Local thus refers to being confined to a subsystem of a larger system, without requiring the subsystem itself to be localized (it can thus itself exist of spatially separated parts)." (See

the first footnote in [8].) We saw in Sections 6 and 7 that the state of a subsystem is 'untouchable' in the sense of distant manipulation only if it is uncorrelated with the complementary cluster. Hence, (20) could pass as a 'locality' condition.

The author derives from (20) a Bell-like inequality with the intention to violate it. Violation of a necessary condition (the inequality) of relation (20) implies violation of (20) itself. I confine my discussion to (20).

Let us return to the singlet state mentioned at the beginning of this section. If it is combined with another bipartite pure or mixed state (it can be combined only by tensor product) into a four-partite system, it will satisfy Seevinck's "local realism" condition as obvious from the UCD theorem (Theorem 2). Hence the spin-projection anti-correlation in the singlet state can be interpreted as local and real.

The author takes a particular state vector

$$\mid\Psi\rangle = (1/2)^{1/2}\Big(\mid\uparrow\downarrow\uparrow\downarrow\rangle - \mid\downarrow\uparrow\downarrow\uparrow\rangle\Big) \tag{21}$$

of a four-partite system. Then he shows that the Bell-like inequality, hence also (20), is violated.

For the validity of (20) it is a relevant question if (21) is homogeneous or heterogeneous. We show now that the former is true.

The tensor-product states $|\uparrow\downarrow\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\downarrow\uparrow\rangle$ are quadri-orthogonal, i. e., orthogonal in each of the four factors. Hence, it is written in a maximally entangled Schmidt canonical form ([3], section 2) how ever one makes the quadri-partite system bipartite. It is thus seen that $\mid\Psi\rangle$ of (21) is not a tensor product of two two-partite subsystems. No single-particle reduced density operator is a ray projector. Hence, (21) cannot contain a single-particle tensor factor state vector.

We conclude in this way that the finest uncorrelated cluster decomposition of (21) is the trivial decomposition. No wonder that (20) does not hold for it.

Now I make a short relevant deviation. As it is well known, the numerous Bell-like inequalities for hidden local values of observables, which were statistical, i. e., applied to ensembles, were superseded by equalities proving Bell's theorem saying that quantum mechanics does not allow an extrapolation with local hidden values of all local observables for *individual systems*. This came later. But as early as in 1977 Stapp has proved an interesting theorem saying that the assumption that individual systems have definite local

values of all local observables is in contradiction with quantum mechanics ([25]).

Mermin has postulated (as part of the Ithaca program [1]) that probabilities should have some physical meaning for individual systems. Hence, Seevinck's plausible Bell-like inequality for correlations makes me conjecture that

(i) if one were able to extend the (vague) quantum correlations idea to a subquantum level, then

(ii) one might be able to derive a Stapp-like theorem saying that correlations in parts of individual clusters that have a homogeneous state in the state of a supersystem could be distantly manipulated from other parts of the cluster as in section 6. In this sense I could agree with Seevinck that individual-system correlations (only their hypothetical subquantum extension) in subclusters of clusters with a homogeneous state lack "robust reality". But this is a bit far fetched.

9.B Cabello's articles

In [6] the author presents the EPR-type disentanglement described in Section 7. He uses this example to attack Mermin's Ithaca program arguing as follows (p. 114, left column).

"Mermin's interpretation assumes *physical locality*, defined as "[t]he fact that the internal correlations of a dynamically isolated system do not depend on any interactions experienced by other systems external to it."" Cabello claims that the mentioned EPR-type disentanglement refutes Mermin's "physical locality". Somewhat below this the author continues his argument (having in mind (18)).

"By this violation of physical locality I do not mean that the internal correlations between particles 1 and 4 "change" after a spacelike separated experiment (this does not happen in the sense that no new internal correlations are "created" that were not "present" in the reduced density matrix for the system 1 and 4 before any interaction), but that the type of internal correlations (and therefore, according to Mermin, *the reality*) of an individual isolated system *can be chosen at distance*." (Only the last italics are due to the present author.)

Now I analyse Cabello's criticism leaning on the present study and on the concrete example of Cabello's mentioned EPR-type disentanglement.

(i) As it was shown in subsection 6.C, no unitary interaction of an external system with subsystem $(2+3)$ can influence the state ρ_{14} of the opposite cluster $(1+4)$. In this sense Mermin's quoted 'physical locality' is satisfied.

(ii) If one interprets the mentioned EPR-type disentanglement in an Everett-like relative-state way (cf Sections 6 and 7), then all that happens when the Bell-states measurement is performed on the subsystem $(2+3)$ is that also the measurement apparatus becomes correlated with subsystem $(1+4)$ (cf an analogous discussion in more detail in [20]). Then again Mermin's point of view is unshaken.

(iii) If one interprets the distant effects of the Bell-states measurement on $(2+3)$ assuming collapse (and modification of the unitary dynamical law), then Mermin's 'physical locality' is refuted. In this sense Cabello's criticism seems justified.

Further, Cabello claims (p.114, second column): "A consistent interpretation (he means the Ithaca program, FH) could be developed by keeping correlations as fundamental but avoiding to say that they are local properties." In view of the mentioned EPR-type correlations, particularly item (iii) above, this claim seems plausible. Though I, for my part, would add to "fundamental" also "real".

Cabello seems partial to the Copenhagen interpretation of quantum mechanics and he says: "Mermin's proposal can be seen as an attempt to complete the Copenhagen interpretation." In contrast to this view, it seems to me that the Ithaca program [1] can be understood as seeing *reality* in the quantum state and *in the correlations* it contains, and not just being "a purely symbolic procedure" according to words of Bohr as Cabello quotes them.

It seems to me that Cabello in his second paper [7] assumes the interpretation with collapse (cf item (iii) above) and elaborates the untenability of 'local' correlations.

10. Conclusion

It was stated in the discussion of Seevinck's article that it seems likely that individual-system correlations in parts of clusters that have a homogeneous state may lack 'robustness of reality' because they can be distantly, i. e., without interaction, 'changed' (in the collapse approach). The point to note

is that this applies to *individual systems.* How ever strongly we wish to understand the physics of individual systems, quantum mechanics is, unfortunately, the physics of *ensembles* of quantum systems. (As well known, even Einstein had no quarrel with this. He only claimed that physics of individual quantum systems will require more than just the quantum state. We do not know if he was right or not.)

The present study along the lines of Mermin's ideas treats correlations only through probabilities, and these are observed ensemblewise. Leaning on this fact, one can claim that the reality of correlations *is locally robust.* In other words, the mentioned manipulation of one subcluster by another (as in Sections 6 and 7) is a global effect, i. e., it takes place in the supercluster (with a homogeneous state). Locally there is no manipulation.

Mermin in the first place and to the largest extent, but also Jordan, Cabello and, particularly, Seevinck began a very intriguing investigation into fundamental many-partite quantum mechanics. I have joined this line of research with the intention to help to lift the mist shrouding the field. Time will tell to what extent our efforts, jointly taken, have achieved this. But I think that we have at least turned the mist into haze; and this, if it lasts, will not be so difficult to disperse.

Appendix

The aim is to characterize *density matrices* in a finite, M-dimensional unitary space. We'll do it, following Wootters [4], representing them in a linearly independent basis consisting of projectors.

To begin with, we study the concept of linear independence.

Lemma The following three properties of a set of Q elements $\{P_q : q = 1, 2, \ldots, Q\}$ of a Q-dimensional unitary space are *equivalent.*

(i) If $\sum_q \omega_q P_q = 0$, then necessarily $\omega_q = 0, \; q = 1, 2, \ldots, Q$.

(ii) Let $\{A_r : r = 1, 2, \ldots, Q\}$ be an orthonormal basis. The transition matrix α , the elements of which appear in

$$P_q = \sum_{r=1}^{Q} \alpha_{qr} A_r, \;\; q = 1, 2, \ldots, Q \qquad (A.1)$$

is non-singular, or equivalently, its determinant is non-zero.

(iii) The matrix $\{\text{tr}\Big(P_q P_{q'}\Big) : q, q' = 1, 2, \ldots, Q\}$ is non-singular, i. e., its determinant is non-zero.

Proof. $\Big($"(ii)" $\Leftrightarrow$ "(iii)"$\Big)$: $\text{tr}\Big(P_q P_{q'}\Big) = \sum_r \sum_{r'} \alpha_{pr} \alpha_{q'r'} \text{tr}\Big(A_r A_{r'}\Big)$. Since by definition $\text{tr}\Big(A_r A_{r'}\Big) = \delta_{rr'}$, one obtains $\text{tr}\Big(P_q P_{q'}\Big) = \sum_r \alpha_{qr} \tilde{\alpha}_{rq'}$, where $\tilde{\alpha}$ is the transpose of α . This implies $det\Big[\text{tr}\Big(P_q P_{q'}\Big)\Big] = \Big(det[\alpha]\Big)^2$. Hence, $det[lhs] \neq 0 \;\Leftrightarrow\; det[rhs] \neq 0$.

$\Big($"(i)" $\Leftrightarrow$ "(ii)"$\Big)$: Let us write down the identity

$$\sum_q \omega_q P_q = \sum_q \omega_q \sum_r \alpha_{qr} A_r = \sum_r \Big(\sum_q \tilde{\alpha}_{rq} \omega_q\Big) A_r, \qquad (A.2)$$

where $\{\omega_q : q = 1, 2, \ldots, Q\}$ are any numbers, and $\{\alpha_{qr} : q, r = 1, 2, \ldots, Q\}$ is the expansion matrix (A.1) (which need not be non-singular at this stage). If we put the identity (A.2) equal to zero, then necessarily also

$$\sum_q \tilde{\alpha}_{rq} \omega_q = 0, \quad r = 1, 2, \ldots, Q \qquad (A.3)$$

is satisfied due to the assumed orthonormality of the basis $\{A_r : r = 1, 2, \ldots, Q\}$.

If we assume the validity of "(i)", then it follows from relation (A.3) that the matrix $\tilde{\alpha}$, hence also α is non-singular, i. e., that "(ii)" is valid. Conversely, if requirement "(ii)" is satisfied, then the matrix $\tilde{\alpha}$ being non-singular, relation (A.3) implies $\omega_q = 0,\ q = 1, 2, \ldots, Q$, i.e., in view of the first expression in (A.2), "(i)" is satisfied. $\square$

Any of the three requirements defines a *linearly independent* sequence $\{P_q : q = 1, 2, \ldots, Q\}$.

We envisage the density operator expanded $\rho = \sum_{q=1}^{M^2} \chi_q P_q$ in M^2 linearly independent projectors. (Now $Q \equiv M^2$.) On the other hand, it is determined by what one can measure, i. e., by the probabilities $\Big\{\text{tr}\Big(P_q \rho\Big) : q = 1, 2, \ldots, M^2\Big\}$. Substituting the expanded form of ρ in the probabilities, one obtains

$$\text{tr}(P_q \rho) = \sum_{q'=1}^{M^2} \chi_{q'} \text{tr}(P_q P_{q'}), \quad q = 1, 2, \ldots, M^2. \qquad (A.4)$$

On account of the non-singularity of the matrix $\{\mathrm{tr}(P_qP_{q'}) : q,q' = 1,2,\dots,M^2\}$ (cf definition (iii) of linear independence), it has an inverse, and the expansion coefficients $\{\chi_q : q = 1,2,\dots,M^2\}$ are uniquely determined.

The number of M^2 , or rather of (M^2-1) probabilities if one takes the identity operator as one of the linearly independent projectors, determines a Hermitian operator (as we have argued in the text). One may wonder how can Wootters be sure that ρ will be a density operator (a non-negative operator), which is a special case of Hermitian operators. My conjecture is that Wootters leaned on Gleason's theorem [2] in this, which guarantees that one must be dealing with a density operator.

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